Functional Analysis Qualifying Exams

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Contents

1 Introduction

This file contains several years (2019-2023) of qualifying exams for functional analysis at Pennsylvania State University. I classified them by the topics. It includes Open Mapping and Closed Graph Theorems, Compact Operators, Spectrum of Operators, Weak Convergence, Hahn-Banach Extensions, Convex Sets and Projection problems. These topics are the ones that appear the most in my opinion. Some other questions are still important but are not included in the solutions. I also attached some basic knowledge at the beginning of every section. These content follows from *Lecture Notes on Functional Analysis With Applications to Linear Partial Differential Equations* by Professor Alberto Bressan. Moreover, solutions for 2024 May Qualifying Exam are also provided. GOOD LUCK :P

2 Open Mapping and Closed Graph Theorems

2.1 Basics

Definition 2.1 (Open and Closed Operators)**.** *Let X and Y be Banach spaces.*

- *$f : X \to Y$ is open if $f(U) \subset Y$ is open for every open subset $U \subset X$.*
- *$f : X \to Y$ is closed if $\{(x, y) : x \in Dom(f), y = f(x)\} \subset X \times Y$ is a closed subset.*

Theorem 2.2 (Open Mapping and Closed Graph Theorems)**.** *Let X and Y be Banach spaces. Let $T : X \to Y$ be a linear operator.*

- *If T is bounded and surjective, then T is open.*
- *If T is closed on X, then T is continuous.*

Corollary 2.3. *If $T : X \to Y$ is a bijective continuous linear operator, then T^{-1} is continuous.*

To prove the continuity/ boundedness of operators, one may consider to use the closed graph theorem.

2.2 Problems

2023 May Q1 Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. Let $T : X \to Y$ be a linear operator such that for any $\lambda \in Y^*, \lambda \circ T \in X^*$. Show that T is a bounded operator, $T \in \mathcal{B}(X, Y)$.

Proof.
To show T is bounded, which is equivalent to showing continuity of T. We may show T is closed, hence continuous. Suppose x_n in X and $x_n \to x$ in X. Let $Tx_n = y_n$ and assume $y_n \to y$. We want to show $T_x = y$. Note that $\lambda \circ T \in X^*$, so $\lambda \circ T$ is continuous. Then

$$\lim_{n\to\infty} (\lambda \circ T)(x_n) = (\lambda \circ T)(\lim_{n\to\infty} x_n) = (\lambda \circ T)x$$

$$\lim_{n\to\infty} (\lambda \circ T)(x_n) = \lim_{n\to\infty} \lambda(y_n) = \lambda(y)$$

If $Tx \neq y$, then $Tx - y \neq 0$. However, for any $\lambda \in Y^*$, we have

$$\lambda(Tx - y) = \lambda(Tx) - \lambda(y) = 0$$

But exists $\lambda \in X^*$ such that $\lambda(\alpha(Tx - y)) = \alpha$ for any $\alpha \in \mathbb{R}$. Then $\lambda(Tx) - \lambda(y) \neq 0$ which contradicts to the above statement. Therefore, $Tx = y$. So, T is closed.

□

2021 May Q4 Let X be $\mathcal{C}([0,1])$ with the supremum norm $\|\cdot\|_\infty$, and let $\Lambda : X \to X$ be a linear operator. Suppose that for any sequence (f_n) in X and any $f \in X$,

$$\|f_n - f\|_1 \to 0 \text{ implies } \|\Lambda(f_n) - \Lambda(f)\|_1 \to 0, \text{ where } \|\cdot\|_1 \text{ is the } L^1 \text{ norm.}$$

Prove that Λ is continuous as a linear operator from X to X.

Proof.
We may prove Λ is closed. Assume $f_n \to f$ on $C[0,1]$ and $\Lambda f_n \to g$ on $C[0,1]$. We first show that $f_n \to f$ on $L^1[0,1]$. Let $\forall \epsilon > 0$, $\exists N > 0$ s.t. $n > N$ implies $\|f_n - f\| < \epsilon$. That is, $\max_{x\in[0,1]}|f_n(x) - f(x)| < \epsilon$. Thus,

$$\int_0^1 |f_n(x) - f(x)| dx \leq \int_0^1 \epsilon dx = \epsilon$$

Therefore, $f_n \to f$ on $L^1[0,1]$. We then conclude that $\Lambda f_n \to \Lambda f$ on $L^1[0,1]$. Similarly, we have $\Lambda f_n \to g$ on $L^1[0,1]$. Then we conclude that $\Lambda f = g$, ie: Λ is closed. By Closed Graph Theorem, we have Λ is continuous. □

2023 Jan Q3 Let Y and Z be vector subspaces of a Banach space X such that every $x \in X$ has a unique representation $x = y + z$ with $y \in Y$ and $z \in Z$. Prove the equivalence of the following statements:

(i) The subspaces Y, Z are closed.

(ii) There exists a constant C such that $C\|x\| \geq \|y\| + \|z\|$ for every $x = y + z \in X$.

Proof.
$(i) \implies (ii)$: Suppose that both Y and Z are closed. Define $T : Y \times Z \to X$ by $T(y,z) = y+z$. Note that $X = Y \oplus Z$, we know for any $x \in X$, exists a unique pair (y,z) such that $T(y,z) = y+z = x$. Therefore, T is a bijective linear map. Also,

$$\|T\| = \sup_{\|y\|+\|z\|\leq 1} \|T(y,z)\| = \sup_{\|y\|+\|z\|\leq 1} \|y+z\| \leq \sup_{\|y\|+\|z\|\leq 1} \|y\|+\|z\| \leq 1$$

T is bounded, hence continuous. So T is a continuous bijection. We know that T^{-1} is continuous by open mapping theorem. Exists $C > 0$ such that $\|T^{-1}\| \leq C$. So for any $x \in X$, we have

$$\|T^{-1}x\| \leq \|T\|\|x\| \leq C\|x\|$$

$$\implies \|y\|+\|z\| \leq C\|x\|$$

$(ii) \implies (i)$: Suppose $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in Y and $y_n \to x$ for some $x \in X$. To prove Y is closed, we want to show that $x \in Y$. Since $X = Y \oplus Z$, there exist unique $y \in Y$, $z \in Z$ such that $x = y + z$. To show $x \in Y$, it is sufficient to show that $z = 0$. Consider the vector $x - y_n \in X$, we know $x - y_n = (y - y_n) + z$. Note that $y - y_n \in Y$ and $z \in Z$. Then

$$C\|x - y_n\| \geq \|y - y_n\| + \|z\|$$

Since $y_n \to x$, we know $C\|x - y_n\| \to 0$. Thus, RHS tends to 0. That is, $\|z\| = 0 \implies z = 0$. Therefore, $x = y \in Y$. Y is closed. Similarly, we can argue that Z is closed. □

2020 Aug Q2 Let X and Y be Banach spaces and let $\Lambda : X \to Y$ be a compact linear operator. Prove that if the range of Λ is closed then the range is finite dimensional.

Proof.
Since $U := Im(\Lambda)$ is closed, U is a closed subspace of a Banach space, we know that U is also a Banach space. Note that compact linear operators are always continuous. By Open Mapping Theorem, we know $\Lambda : X \to U$ is open. Then $\overline{\Lambda(B_X(0,1))}$ is compact. We conclude that $Im(\Lambda)$ is finite-dimensional.

Remark: The converse statement is also true.

Proof.
This is just the consequence of every finite-dimensional linear subspace of a normed vector space over $\mathbb{R}$ or $\mathbb{C}$ is also closed. □

Additional Remark: The range of compact operators can be approximated by a finite-dimensional subspace. That is, $\forall \varepsilon > 0, \exists$ a subspace $U \subset Im(\Lambda)$ such that

$$\inf_{s \in U} \|\Lambda x - s\| \leq \varepsilon \|x\|$$

□

3 Hahn-Banach Extensions

3.1 Basics

Theorem 3.1 (Hahn-Banach Extension)**.** *Suppose a vector space X over $\mathbb{R}$ and $p : X \to \mathbb{R}$ with the following properties:*

(a) $p(x+y) \leq p(x) + p(y), \ \forall x, y \in X$

(b) $p(tx) = tp(x), \ \forall t \geq 0$

If a subspace $V \subset X$ with a linear functional $f : V \to \mathbb{R}$ such that

(c) $f(x) \leq p(x), \ \forall x \in V$

Then exists a linear funcitonal $F : X \to \mathbb{R}$ such that $F(x) = f(x)$ for any $x \in V$ and $-p(-x) \leq F(x) \leq p(x)$ for all $x \in X$.

Theorem 3.2 (Extension Theorem for Bounded Linear Functionals)**.** *Let X be a normed space over $\mathbb{R}$ and $f : V \to \mathbb{R}$ be a bounded linear functional defined on a subspace $V \subset X$. Then f can be extended to a bounded linear functional $F : X \to \mathbb{R}$ with $\|F\| = \|f\|$.*

Theorem 3.3 (Separation of Convex Sets)**.** *Suppose $X_{\mathbb{R}}$ is a normed space. Let $A, B \subset X$ and $A \cap B = \phi$. Then*

- *If A is open, then exists a bounded linear functional $f : X \to \mathbb{R}$ and $c \in \mathbb{R}$ such that*

$$\phi(A) < c \leq \phi(b), \ \forall a \in A, b \in B$$

- *If A is compact and B is closed, then exists a bounded linear functional $f : X \to \mathbb{R}$ and $c_1, c_2 \in \mathbb{R}$ such that*

$$\phi(A) \leq c_1 < c_2 \leq \phi(b), \ \forall a \in A, b \in B$$

3.2 Problems

2024 Jan Q3 Prove that there exists a bounded linear functional $\Lambda : L^\infty(\mathbb{R}) \to \mathbb{R}$ with $\|\Lambda\| = 2$ such that $\Lambda f = f(1) - f(0)$ for every bounded continuous function f.

Proof.
Note that $BC(\mathbb{R})$ is a subspace of $L^\infty(\mathbb{R})$. Let $T : BC(\mathbb{R}) \to \mathbb{R}$ be a linear functional defined by $Tf = f(1) - f(0)$. Then

$$\|T\| = \sup_{\|f\|=1} \|Tf\| = \sup_{\|f\|=1} |f(1) - f(0)| \leq \sup_{\|f\|=1} (|f(1)| + |f(0)|) = 2$$

Take

$$f(x) = \begin{cases} 1 & \text{, if } x > 1 \\ 2x - 1 & \text{, if } 0 \leq x \leq 1 \\ -1 & \text{, if } x < 0 \end{cases}$$

We know $f \in BC(\mathbb{R})$, $\|f\| = 1$ and $\|Tf\| = 2$,. Thus, $\|T\| = 2$. By Hahn-Banach Extension, there exists $\Lambda : L^\infty(\mathbb{R}) \to \mathbb{R}$ such that $\Lambda f = Tf$ for any $f \in BC(\mathbb{R})$ and $\|\Lambda\| = 2$. □

2023 May Q3 Prove that there exists a bounded linear functional Λ on $L^\infty(\mathbb{R})$, with norm $\|\Lambda\| \leq 2$, such that

$$\Lambda(f) = \lim_{t \to +\infty} \frac{1}{t} \int_{-t}^{t} f(x) \sin(x) dx$$

for all $f \in L^\infty(\mathbb{R})$ for which the limit exists.

However, show that this functional cannot be represented in the form

$$\Lambda(f) = \int_{-\infty}^{\infty} \phi(x) f(x) dx,$$

for any $\phi \in L^1(\mathbb{R})$.

Proof.
Define

$$U := \{f : L^\infty(\mathbb{R}) \to \mathbb{R} \text{ where } \lim_{t \to \infty} \frac{1}{t} \int_{-t}^{t} f(x) \sin(x) dx \text{ exists}\}$$

Suppose $f_1, f_2 \in U$, then $f_1 + f_2 \in U$. Also $cf_1 \in U$ for any $c \in \mathbb{R}$. So U is a subspace of $L^\infty(\mathbb{R})$. Define

$$T(f) := \lim_{t \to +\infty} \frac{1}{t} \int_{-t}^{t} f(x) \sin(x) dx$$

Then

$$\|T\| = \sup_{\|f\|=1} \|Tf\| \leq \lim_{t \to +\infty} \frac{1}{t} \int_{-t}^{t} |f(x) \sin(x)| dx \leq \lim_{t \to +\infty} \frac{1}{t} \int_{-t}^{t} |\sin(x)| dx \leq \lim_{t \to \infty} \frac{2t}{t} = 2$$

By Hahn-Banach Extension, we know exists $\Lambda : L^\infty(\mathbb{R}) \to \mathbb{R}$ that satisfies the condition. Consider function $f(x) = \chi_{[a,b]}(x)$. Then

$$\lim_{t \to +\infty} \frac{1}{t} \int_{-t}^{t} f(x) \sin(x) dx = \lim_{t \to \infty} \int_{a}^{b} \sin(x) dx = 0$$

Suppose that the statement in question is true, then for any $\phi \in L^1$, we have

$$\int_{-\infty}^{\infty} \phi(x) f(x) dx = 0$$

Then $\phi(x) = 0$ for all intervals $[a, b]$. Thus,

$$\lim_{t \to +\infty} \frac{1}{t} \int_{-t}^{t} f(x) \sin(x) dx = 0 \quad \text{for all } f \in L^\infty(\mathbb{R})$$

However, one may come up with a counterexample, ie: $f(x) = \sin x$, then $\Lambda f \neq 0$. □

May 2022 Q1 Prove that there exists a bounded linear functional Φ on the space $L^\infty(\mathbb{R})$, with $\|\Phi\| = 1$, such that

$$\Phi(f) = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_0^\epsilon f(x) dx$$

whenever the above limit exists.

On the other hand, show that there does not exist any function $\phi \in L^1(\mathbb{R})$ such that

$$\Phi(f) = \int_{-\infty}^{+\infty} \phi(x) f(x) dx, \text{ for all } f \in L^\infty(\mathbb{R}).$$

Conclude that $L^1(\mathbb{R})$ is not the dual space of $L^\infty(\mathbb{R})$.

Proof.
Define

$$U := \{f : L^\infty(\mathbb{R}) \to \mathbb{R} \text{ where } \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_0^\epsilon f(x) dx \text{ exists}\}$$

Then U is a subspace of $L^\infty(\mathbb{R})$. Define a linear functional $T : U \to \mathbb{R}$ by

$$T(f) := \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_0^\epsilon f(x) dx$$

We have

$$\|T\| = \sup_{\|f\|=1} \|Tf\| = \sup_{\|f\|=1} \left| \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_0^\epsilon f(x) dx \right| \leq \sup_{\|f\|=1} \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_0^\epsilon |f(x)| dx \leq 1$$

Take $f \equiv 1$, then $\|Tf\| = 1$. So $\|T\| = 1$. By Hahn-Banach Extension, we have the first part done. Assume exists $\phi(x) \in L^1(\mathbb{R})$ such that the expression holds. For any $a > 0$, let $f(x) = \chi_{(0,a]}(x)$. Then

$$\Phi(f) = \lim_{\epsilon \to 0^+} \int_0^\epsilon f(x) dx = 0 = \int_0^a \phi(x) dx$$

Then $\phi(x) = 0$ for a.e. $x > 0$. Similarly, we can prove that $\phi(x) = 0$ for a.e. $x < 0$. So $\phi(x) = 0$ almost everywhere on $\mathbb{R}$. Thus, $\Phi(f) = 0$. However, $\Phi(f) = 1$ for $f \equiv 1$. So we conclude that the expression cannot hold. So $(L^\infty(R))^*$ is strictly larger than $L^1(\mathbb{R})$. □

2021 May Q1 Let X be a Banach space, let Y be its closed subspace, and let $a \in X \setminus Y$. Prove that there exists a bounded linear functional $F : X \to \mathbb{R}$ with $\|F\| = 1$ such that

$$F(y) = 0 \text{ for all } y \in Y \text{ and } F(a) = d(a, Y) = \inf_{y \in Y} \|a - y\|$$

Proof.
Define $V = Y \oplus span\{a\}$. The direct sum is secured since $a \in X \backslash Y$. We want to construct a linear function $\phi : V \to \mathbb{R}$ such that

$$\phi(y) = 0 \text{ for all } y \in Y \text{ and } \phi(a) = d(a, Y) = \inf_{y \in Y} \|a - y\|$$

Define $\phi : V \to \mathbb{R}$ by

$$\phi(v) = r \inf_{y \in Y} \|a - y\|$$

where $v = u + ra$ with $u \in Y$ and $r \in \mathbb{R}$. To show $\|\phi\| = 1$, we first show that $\|\phi\| \leq 1$. It is equivalent to show

$$r \leq \frac{1}{\inf_{y \in Y} \|a - y\|}$$

Note that $\inf_{y \in Y} \|a - y\| > 0$ follows from the closedness of the subspace Y. For any $\|v\| \leq 1$, we have

$$\begin{aligned} \|v\| \leq 1 &\implies \|u + ra\| \leq 1 \\ &\implies r \|\frac{u}{r} + a\| \leq 1 \\ &\implies r \leq \frac{1}{\|\frac{u}{r} + a\|} \end{aligned}$$

Take $w = \frac{-u}{r}$, then

$$r \leq (\|a - w\|)^{-1} \leq \left(\inf_{y \in Y} \|a - y\|\right)^{-1}$$

Therefore, $\|\phi\| \leq 1$. Since $\inf_{y \in Y} \|a - y\|$ can be approximated by a sequence, let $y' \in Y$ be the limit point. Take

$$v = \frac{-y' + a}{\|-y' + a\|}$$

Then $\|v\| = 1$ and

$$\phi(v) = \frac{1}{\|-y' + a\|} \times \inf_{y \in Y} \|a - y\| = 1$$

Thus, $\|\phi\| = 1$. By Hahn-Banach Extension, we know exists $F : X \to \mathbb{R}$ such that

$$F(v) = \phi(v) \text{ for all } v \in V \text{ and } \|F\| = \|\phi\| = 1$$

□

4 Compact and Self-Adjoint Operators

4.1 Basics

Definition 4.1 (Compact Operator). *Let X, Y be Banach spaces. We say T is compact if for any bounded sequence $\{x_n\}$ in X, we have a convergent subsequence $\{T_{x_{n_k}}\}$ in Y. That is, for any bounded subset $U \subseteq X$, $\overline{T(U)}$ is compact.*

Theorem 4.2. *Let X and Y be Banach spaces.*

- *Let $T \in \mathcal{B}(X, Y)$. Then if $Im(T)$ is finite-dimensional, then T is compact.*

- *Let $T_n : X \to Y$ be a sequence of compact operators. If*

$$\lim_{n\to\infty} \|T_n - T\| = 0$$

then T is also compact.

Theorem 4.3. *Let $T \in \mathcal{B}(X, Y)$, where X and Y are Banach. Then T is compact iff T^* is compact.*

To prove the compactness of an operator, Arzelà-Ascoli Theorem is crucial.

4.2 Problems

2024 Jan Q1 Let $A : H \to H$ be a bounded, self-adjoint operator on the Hilbert space H. Prove that the following are equivalent:

(i) Range(A) is dense.

(ii) Ker(A) = $\{0\}$.

Proof.
Suppose $u \in Ker(A^*)$, for any $v \in H$, we have

$$\langle u, Av \rangle = \langle A^* u, v \rangle = 0 \Longrightarrow u \in (Range(A))^\perp$$

So $Ker(A^*) \subseteq (Range(A))^\perp$.
Suppose $u \in (Range(A))^\perp$, then for any $v \in H$, we have

$$\langle u, Av \rangle = \langle A^* u, v \rangle = 0 \Longrightarrow A^* u = 0 \Longrightarrow u \in Ker(A^*)$$

So $(Range(A))^\perp \subseteq Ker(A^*)$. Thus, $(Range(A))^\perp = Ker(A^*)$. Since A is self-adjoint, we have $Range(A)^\perp = Ker(A)$. Then the statement is trivial. □

2024 Jan Q4 Consider the operator

$$(\Lambda x)(s) := \int_{-1}^{1} s^2 t^2 x(t) dt$$

from $L^2([-1, 1])$ into itself. Prove that Λ is compact and find its spectrum.

Proof.
One may prove the compactness by using Arzelà-Ascoli Theorem, which is similar to the questions below. Aside from that, notice that

$$(\Lambda x)(s) = s^2 \int_{-1}^{1} t^2 x(t) dt = s^2 (\Lambda x)(1)$$

Since $L^2[-1, 1]$ is an infinite-dimensional Hilbert space and Λ is compact, we know $\sigma(\Lambda) = \sigma_p(K) \cup \{0\}$. Let $\lambda \neq 0$ be an eigenvalue of Λ. Then

$$(\Lambda x)(s) = s^2 (\Lambda x)(1) = \lambda x(s)$$

$$\Longrightarrow s^2 \lambda x(1) = \lambda x(s) \Longrightarrow x(s) = x(1) s^2$$

Then we have

$$(\Lambda x)(s) = s^2 \int_{-1}^{1} t^2 x(t) dt = s^2 \int_{-1}^{1} t^2 \cdot t^2 x(1) dt = \frac{2}{5} x(1) s^2$$

So

$$(\Lambda x)(s) = \frac{2}{5}x(1)s^2 = \lambda x(s) = \lambda x(1)s^2 \Longrightarrow \lambda = \frac{2}{5}$$

Therefore, $\sigma(\Lambda) = \{0, \frac{2}{5}\}$. □

2023 May Q2 Consider the Volterra operator:

$$(Kf)(x) = \int_0^x f(y)dy$$

on $L^2([0,1])$. Prove that K is compact, that Range$(K) \subset \{u \in \mathcal{C}([0,1]) | u(0) = 0\}$, and that the spectral radius r_K of K equals 0.

Proof.
K is compact $\Longleftrightarrow$ for any bounded sequence $\{f_n\}$ in $L^2[0,1]$, exists a subsequence $\{Kf_{n_k}\}$ of $\{Kf_n\}$ converges uniformly. By using Arzelà-Ascoli Theorem, it suffices to show $\{Kf_n\}$ bounded uniformly and is equicontinuous.

Obviously, K is a linear operator. Let $g(x,y) = \chi_{[0,x]}(y)$. Then $(Kf)(x) = \int_0^1 g(x,y)f(y)dy$. Then

$$\|fg\|_1 \leq \|f\|_2\|g\|_2 = \sqrt{x}\|f\|_2$$

We have

$$\|K\| = \sup_{\|f\|=1} \left\| \int_0^x f(y)dy \right\| = \sup_{\|f\|=1} \left(\int_0^1 \left(\int_0^x f^2(y)dx \right) \right)^{\frac{1}{2}} \leq \sup_{\|f\|=1} \left(\int_0^1 \left(\sqrt{x}\|f\|_2 \right)^2 dx \right)^{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

For any bounded sequence $\{f_n\}$ in $L^2[0,1]$, let $\|f_n\| \leq M$ for all $n \in \mathbb{N}$. Then

$$\|Kf_n\| \leq \|K\|\|f_n\| \leq \frac{1}{\sqrt{2}}M$$

Remark: However, uniform boundedness in norm does not implies uniform boundedness for all elements in [0,1]. For Asrzelà-Ascoli Theorem, we are required the uniform boundedness for elements in [0,1]. For any $x \in [0,1]$, we have

$$|Kf_n(x)| = \left| \int_0^x f(y)dy \right| \leq \int_0^x |f(y)|\, dy \leq \int_0^1 |f(y)|dy \leq \|f_n\|_2\|1\|_2 \leq M$$

Thus, $\sup_{x\in[0,1]}|Kf_n(x)| \leq M$ for all $n \in \mathbb{N}$. WLOG, suppose $\times' \leq x$ in $[0,1]$ and $|x' - x| < \delta = \frac{\epsilon}{M}$. Then

$$|Kf_n(x) - Kf_n(x')| = \left| \int_{x'}^x f(y)dy \right| = \left| \int_0^1 f(y)\chi_{[x',x]}(y)dy \right| \leq \|f_n\|_2\|\chi_{[x',x]}\|_2 \leq M\delta < \epsilon$$

Thus, we conclude that $\{Kf_n\}$ is equicontinuous. By Arzelà-Ascoli, we know K is a compact operator. By similar argument, we know $Kf = u$ is continuous on $[0,1]$. $u(0) = \int_0^0 f(y)dy = 0$.

Since K is a compact operator on $L^2[0,1]$, an infinite-dimensional Hilbert space, we know $0 \in \sigma(K)$ and $\sigma(K) = \sigma_p(K) \cup \{0\}$. Assume there exists a nonzero eigenvalue $\lambda \in \mathbb{R}$. Then

$$Kf = \lambda f \Longrightarrow (Kf)' = f = \lambda f' \Longrightarrow f(x) = C \cdot e^{\frac{x}{\lambda}}$$

However, $Kf(0) = 0$ implies $C\frac{1}{\lambda}e^0 = 0$, which means $C = 0$. Then the eigenvector $f = 0$, which is a contradiction. Then we conclude that there is no nonzero eigenvalue, $r_K = 0$. □

2023 Jan Q2 Let $\varphi : [0,1] \to \mathbb{R}$ be a continuous function. Define the linear operator K on $L^2([0,1])$ by setting

$$(Kf)(x) = \int_0^x \varphi(y)f(y)dy.$$

Prove that K is compact and find its adjoint.

Proof.
Since $\varphi \in BC([0,1],\mathbb{R})$, we know $\exists M > 0$ such that $\sup_{x\in[0,1]}|\varphi(x)| \leq M$. For any bounded $\{f_n\}_{n\in\mathbb{N}}$, suppose $\|f_n\| \leq L$ for some $L > 0$. Then for any $x \in [0,1]$,

$$|Kf_n(x)| = \left|\int_0^x \varphi(y)f_n(y)dy\right| \leq \int_0^1 |\varphi(y)f_n(y)|dy \leq M\int_0^1 |f_n(y)|dy \leq M\|f_n\|_2 \leq ML$$

Thus, $\{Kf_n\}$ is uniformly bounded. $\forall\epsilon > 0$, $\exists\delta = \frac{\epsilon^2}{M^2L^2}$ such that for any $x - x' < \delta$, we have

$$|Kf_n(x) - Kf_n(x')| \leq \int_{x'}^x |\varphi(y)f(y)|dy < \epsilon$$

where the last inequality follows from Hölder's inequality. That is,

$$\int_{x'}^x |\varphi(y)f(y)|dy \leq \left(\int_{x'}^x \varphi^2(y)dy\right)^{\frac{1}{2}} \left(\int_{x'}^x f_n^2(y)dy\right)^{\frac{1}{2}} \leq (x - x')^{\frac{1}{2}}ML < \epsilon$$

We have proved that $\{Kf_n\}$ is equicontinuous. By Arzelà-Ascoli Theorem, K is compact.

To find the adjoint of K, we use the definition $\langle Kf, g\rangle = \langle f, K^*g\rangle$. Then

$$(K^*g)(x) = \int_x^1 \varphi(x)g(y)dy$$

□

2022 May Q4 Let $\phi : [0,1] \to \mathbb{R}$ be a positive, strictly increasing continuous function. On the Hilbert space $L^2([0,1])$, consider the linear operator Λ defined by

$$(\Lambda f)(x) = \phi(x)f(x).$$

- Check whether Λ is (i) self-adjoint, or (ii) compact.
- Denoting by $M := \|\Lambda\|$ the operator norm, prove that

$$M := \max_{x\in[0,1]} \phi(x) = \phi(1).$$

Show that M is in the essential spectrum of Λ, but not in the point spectrum.

Proof.
For any $f, g \in L^2[0,1]$, we have

$$\langle \Lambda f, g\rangle = \int_0^1 \phi(x)f(x)g(x)dx = \int_0^1 f(x)(\Lambda^* g)(x)dx$$

So $(\Lambda^*g)(x) = \phi(x)g(x)$. Then $\Lambda = \Lambda^*$. Note that

$$\|\Lambda\| = \sup_{\|f\|=1} \|\Lambda f\| = \sup_{\|f\|=1} \left(\int_0^1 \phi^2(x)f^2(x)dx\right)^{\frac{1}{2}} \leq \sup_{\|f\|=1} \phi(1)\|f\| = \phi(1)$$

where $\sup_{x\in[0,1]}|\phi(x)| \leq \phi(1)$ due to the positivity and strictly increasing properties of ϕ. Let

$$f_n(x) = \sqrt{n}\chi_{[1-\frac{1}{n}]}(x)$$

We can see that $\|\Lambda f_n\| \to \phi(1)$ as $n \to \infty$. Therefore,

$$M = \|\Lambda\| = \phi(1)$$

Remark: We are showing $\|\Lambda\| = \phi(1)$ here. In the question M was defined twice weirdly. It is obvious that $\max_{x\in[0,1]} \phi(x) = \phi(1)$.

Since Λ is self-adjoint, we know $M = \max\{|r|, R\}$, where r and R are

$$r = \inf_{\|f\|=1} \langle \Lambda f, f\rangle$$

$$R = \sup_{\|f\|=1} \langle \Lambda f, f\rangle$$

Also, r and R are in the spectrum of Λ. Notice that $R > r > 0$, which follows from the positivity of ϕ. Therefore, $M = R \in \sigma(\Lambda)$. Now we show that $M \notin \sigma_p(\Lambda)$. Assume $(MI - \Lambda)$ is not injective, ie: $\exists f \in L^2[0,1]$ such that $(\phi(x) - M)f(x) = 0$ for all $x \in [0,1]$. Then $f = 0$, which is not an eigenvector. Therefore, $(MI - \Lambda)$ is injective. So $M \notin \sigma_p(\Lambda)$.

Since $M > 0$ and $M \in \sigma(\Lambda)\backslash\sigma_p(\Lambda)$, we know that Λ cannot be a compact operator. This is because $\sigma(K) = \sigma_p(K) \cup \{0\}$ for any compact operator K on an infinite dimensional Hilbert space.

□

2021 May Q3 Let Λ be a compact linear operator from a Banach space X to a Hilbert space H. Prove that for every $\epsilon > 0$, there exists a linear operator $\Lambda_\epsilon : X \to H$ with finite-dimensional range such that $\|\Lambda - \Lambda_\epsilon\| \leq \epsilon$.

Proof.
Recall another definition of compact operator: $\Lambda : X \to Y$ is compact if and only if $\overline{\Lambda(U)}$ is compact for every bounded $U \subset X$. Therefore, $\overline{\Lambda(B_X(0,1))}$ is compact, denote it as C. By definition of compactness, we know

$$C \subseteq \bigcup_{k=1}^{N} B_H(y_k, \frac{\varepsilon}{2}) \quad \text{for some } y_k \in H$$

Choose $x_k \in B_X(0,1)$ such that $\Lambda x_k = y_k$. Then for any x in the unit ball of X,

$$\|\Lambda x - \Lambda_\varepsilon x\| \leq \|\Lambda x - y_i\| + \|y_i - \Lambda_\varepsilon x\| \leq \frac{\varepsilon}{2} + \|y_i - \Lambda_\varepsilon x\|$$

for some $y_i \in \{y_1, \cdots, y_N\}$. We want to construct a linear operator Λ_ε with finite rank such that $\|y_i - \Lambda_\varepsilon x\| < \frac{\varepsilon}{2}$. Denote $U = span\{y_1, \cdots, y_N\}$, then $H = U \oplus U^\perp$. Define $\Lambda_\varepsilon = P \circ \Lambda$, where P is the orthogonal projection from H onto U. Then

$$\|y_i - \Lambda_\varepsilon x\| = \|y_i - P(\Lambda x)\| \leq \|y_i - \Lambda x\| < \frac{\varepsilon}{2}$$

Thus, $\|\Lambda - \Lambda_\varepsilon\| < \varepsilon$.

Remark: This is the approximation property of compact operators on Hilbert Space.

□

2020 Aug Q3 Let Λ be a bounded linear operator on a separable infinite-dimensional Hilbert space H, and let Λ^* be the adjoint of Λ. Prove that if $\Lambda^*\Lambda$ is compact then Λ is compact.

Proof.
Let $\{x_n\}$ be a bounded sequence in H with $\|x_n\| \leq M$ for some $M > 0$. Then exists a subsequence of $\{\Lambda^*\Lambda x_n\}$, $\{\Lambda^*\Lambda x_{n_k}\}$, converges in H. $\forall \varepsilon > 0, \exists K > 0$ such that $\forall k, j > N$, we have

$$\|\Lambda^*\Lambda x_{n_k} - \Lambda^*\Lambda x_{n_j}\| < \varepsilon$$

Also,

$$\|\Lambda x_{n_k} - \Lambda x_{n_j}\|^2 = \langle \Lambda x_{n_k} - \Lambda x_{n_j}, \Lambda x_{n_k} - \Lambda x_{n_j} \rangle = \langle \Lambda^*\Lambda x_{n_k} - \Lambda^*\Lambda x_{n_j}, x_{n_k} - x_{n_j} \rangle \leq M\varepsilon$$

So $\{\Lambda x_{n_k}\}$ converges. We conclude that Λ is compact. □

2019 May Q2 Consider the space $\ell := \left\{x = (x_1, x_2, \ldots), \|x\| = \sqrt{\sum_i |x_i|^2} < +\infty\right\}$ of square summable sequences of real numbers, with inner product $(x, y) = \sum_i x_i y_i$. Prove that the linear operator $\Lambda : \ell^2 \mapsto \ell^2$ defined by

$$\Lambda(x) = \left(\frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \ldots\right)$$

is compact. Decide whether Λ is (i) surjective and (ii) self-adjoint.

5 Fredholm's Theorem

5.1 Basics

Theorem 5.1 (Fredholm's Theorem)**.** *Let $K : H \to H$ be a compact linear operator, where H is a Hilbert space over $\mathbb{R}$. Then*

- $dim(Ker(I - K)) < \infty$.
- $Im(I - K)$ *is closed.*
- $Im(I - K) = Ker(I - K^*)^\perp$.
- $Ker(I - K) = \{0\} \iff Im(I - K) = H$.
- $dim(Ker(I - K)) = dim(Ker(I - K^*))$.

5.2 Problems

2022 May Q2 Let $K = K(x, y)$ be a measurable function such that $|K(x, y)| \leq 1$ for all $x, y \in [0, T]$. Consider the linear operator $\Lambda : L^2([0, T]) \to L^2([0, T])$ defined by

$$(\Lambda u)(x) = \int_0^T K(x, y)u(y)dy, u \in L^2([0, T]).$$

Assuming that $T < 1$, prove that for every $f \in L^2[(0, T)]$, the equation

$$u + \Lambda u = f$$

has a unique solution $u \in L^2([0, T])$, with $\|u\|_{L^2} \leq \frac{1}{1-T}\|f\|_{L^2}$.

Proof.

$$u + \Lambda u = f \Longrightarrow (I + \Lambda)u = f \Longrightarrow (I - S)u = f, \quad S = -\Lambda$$

Before using Fredholm's theorem, We want to show that Λ is a compact operator. Obviosuly, Λ is a linear operator. Note that $K(x, y)$ is bounded on $[0, T] \times [0, T]$, we know there exists a sequence of continuous functions $K_n(x, y)$ such that $K_n \to K$ pointwise. By Lusin's Theorem, we know $\|K - K_n\| \to 0$. So for sufficient large n, we have $\|K - K_n\| < \frac{\varepsilon}{T}$. Denote $(\Lambda_n u)(x) = \int_0^T K_n(x, y)u(y)dy$. Then

$$\|\Lambda u - \Lambda_n u\| = \left\| \int_0^T (K(x, y) - K_n(x, y)u(y))dy \right\| \leq \int_0^T \|(K(x, y) - K_n(x, y)u(y))\| dy \leq T \cdot \frac{\varepsilon}{T} \cdot \|u\|$$

So for sufficient large n, we know

$$\|\Lambda - \Lambda_n\| \leq \varepsilon$$

Thus, Λ is a compact operator. This follows from the fact that $\{\Lambda_n\}$ is a sequence of compact linear operators, which can be checked by Arzelà-Ascoli Theorem. To show that there exists a unique solution for u, it suffices to show that $I + K$ is bijective. By Fredholm's Theorem, we only need to show that $I + \Lambda$ is injective. It suffices to show that $Ker(I + K) = \{0\}$. Note that $\Lambda 0 = 0$. Also,

$$\|\Lambda\| = \sup_{\|u\|=1} \|\Lambda u\| = \sup_{\|u\|=1} \left\| \int_0^T K(x, y)u(y)dy \right\| < 1$$

following from the fact that $T < 1$. Therefore, $-\Lambda$ is a contraction map. By contraction mapping theorem, we know 0 is the unique fixed point of Λ. So $Ker(I + \Lambda) = \{0\}$. Therefore,

$$u = (I + \Lambda)^{-1} f$$

is unique. Furthermore,

$$\|u\| = \|f - \Lambda u\| \leq \|f\| + \|\Lambda\| \|u\| \leq \|f\| + \sqrt{T} \|u\| \Longrightarrow \|u\| \leq \frac{1}{1 - \sqrt{T}} \|f\|$$

Remark: It turned out that someone writing $\sqrt{T}$ instead of T still received full marks.

□

6 Weak Convergence

6.1 Basics

Definition 6.1 (Dual Space). *Let X be a Banach space over $\mathbb{R}$.*

- *Dual space $X^* := \{\phi : X \to \mathbb{R}$ is a continuous linear functional$\}$*
- $X^* = \mathcal{B}(X, \mathbb{R}),\ \|\phi\|_* = \sup_{\|x\| \leq 1} |\phi(x)|$

Definition 6.2 (Weak Convergence). $x_n \rightharpoonup x$ *if*

$$\lim_{n \to \infty} \phi(x_n) = \phi(x),\ \forall \phi \in X^*$$

Definition 6.3 (Weak-star Convergence). $\phi_n \overset{*}{\rightharpoonup} \phi \in X^*$ *if*

$$\lim_{n \to \infty} \phi_n(x) = \phi(x),\ \forall\ x \in X$$

Comparision:

$$\phi_n \xrightharpoonup{*} \phi \Longrightarrow |\phi_n(x) - \phi(x)| \to 0, \forall x \in X$$

$$\phi_n \to \phi \Longrightarrow \sup_{\|x\|\leq 1} |\phi_n(x) - \phi(x)| \to 0$$

Theorem 6.4 (Banach-Alaoglu)**.** *Let X be a separable Banach space. Then every bounded sequence $\{\phi_n\}$ in X^* admits a weak star convergent subsequence.*

Corollary 6.5. *Let X be a Banach space. Any sequence $x_n \in X$ which converges weakly to some $x \in X$ is necessarily bounded.*

6.2 Problems

2023 Jan Q1 Let H be a Hilbert space.

(i) Prove that every decreasing sequence of bounded, closed, convex, nonempty sets $\Omega_1 \supseteq \Omega_2 \supseteq \dots$ has nonempty intersection.

(ii) Taking $H = L^2([0,1])$, define

$$\Omega_n = \left\{ f \in H : \int_0^1 f(x)dx = 1, f(x) \geq 0 \text{ for all } x \in [0,1], f(x) = 0 \text{ for } x \in \left[\frac{1}{n}, 1\right] \right\}.$$

Compute $\cap_{n\geq 1}\Omega_n$. Expain why this example does not contradict the previous result.

Proof.

(i) Let $x_n \in \Omega_n$ for $n \in \mathbb{N}$. Note that Ω_1 is bounded and $\{\Omega_n\}$ is a decreasing sequence of sets, so $\{x_n\}$ is also bounded. Every bounded sequence has a weak convergent subsequence in a Hilbert space. Therefore, exists $\{x_{n_k}\}$ weakly converges. Let $x_{n_k} \rightharpoonup x$. Also, closed convex sets are weakly closed. So $x \in \Omega_n$ for all $n \in \mathbb{N}$. Thus,

$$x \in \bigcap_{n=1}^{\infty} \Omega_n$$

(ii) Suppose exists $g \in \bigcap_{n=1}^{\infty} \Omega_n$. Then $g(x) = 0$ on $\left[\frac{1}{n}, 1\right]$ for any $n \in \mathbb{N}$. Then $g(x) = 0$ on $(0,1]$, which implies $\int_0^1 g(x)dx \neq 1$. So $g \notin \bigcap_{n=1}^{\infty} \Omega_n$. We conclude that

$$\bigcap_{n=1}^{\infty} \Omega_n = \phi$$

(i) fails because Ω_n is not a bounded set. For example, for any $k \in \mathbb{N}$ and $k \geq n$, let

$$f_k(x) = \chi_{[\frac{1}{k},1]}(x)$$

then $f_k \in \Omega_n$ but $\|f_k\|_2 = k$ is not bounded.

□

2023 Jan Q4 Let $\{\mathbf{e}_n\}$ be a complete orthonormal basis for a Hilbert space H. Let $(a_n)_{n\geq 1}$ be a bounded sequence of numbers, and define $u_n = \frac{1}{n}\sum_{i=1}^{n} a_i \mathbf{e}_i$.

(i) Prove the strong convergence $u_n \to 0$

(ii) Prove the weak convergence $\sqrt{n}u_n \rightharpoonup 0$.

Proof.

(i) Let $|a_n|\leq M$ for all $n \in \mathbb{N}$, where $M > 0$. Then

$$\|u_n\|^2 = \|\sum_{i=1}^{n} \frac{a_i}{n} e_i\|^2 = \sum_{i=1}^{n} \|\frac{a_i}{n} e_i\|^2 = \sum_{i=1}^{n} |\frac{a_i}{n}|^2 \leq \frac{M^2}{n} \longrightarrow 0$$

Therefore, u_n converges to 0 strongly.

(ii) For any $g \in H$, $g = \sum_{k=1}^{\infty} \langle g, e_k \rangle e_k$. Define $\langle g, e_k \rangle = b_k$ for all $k \in \mathbb{N}$.

$$| < \sqrt{n}u_n, g\rangle| = \left|\sum_{i=1}^{n} \frac{1}{\sqrt{n}} a_i b_i\right| \leq \sum_{i=1}^{n} \left|\frac{1}{\sqrt{n}} a_i b_i\right| \leq M \sum_{i=1}^{n} \left|\frac{b_i}{\sqrt{n}}\right|$$

Note that for any $n \in \mathbb{N}$, by Cauchy-Schwarz and Bessel's inequalities

$$\left(\sum_{i=1}^{n} \left|\frac{b_i}{\sqrt{n}}\right|\right)^2 \leq \frac{1}{n} \left(\sum_{i=1}^{n} 1\right) \left(\sum_{i=1}^{n} |b_i|^2\right) \leq \|g\|$$

So $\sum_{i=1}^{n} \frac{|b_i|}{\sqrt{n}}$ converges. Then $\frac{M}{\sqrt{n}} \sum_{i=1}^{n} |b_i| \longrightarrow 0$ Therefore, for any $g \in H$,

$$\lim_{n\to\infty} \langle \sqrt{n}u_n, g \rangle = 0$$

So $\sqrt{n}u_n \rightharpoonup 0$.

□

2022 May Q3 Let $(f_n)_{n\geq1}$ be a sequence of functions in $L^2(\mathbb{R})$. Prove that the weak convergence $f_n \rightharpoonup 0$ in $L^2(\mathbb{R})$ holds if and only if the norms $\|f_n\|_{L^2}$ are uniformly bounded and

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = 0, \text{ for every } a < b.$$

Proof.
($\Longleftarrow$): Since simple function set is dense in $L^2(\mathbb{R})$, we can consider any simple function $g(x) = \sum_{i=1}^{N} \chi_{[\alpha_i, \beta_i]}(x)$. We know

$$\langle f_n, g \rangle = \int_{\mathbb{R}} f_n(s) \chi_{[\alpha_i, \beta_i]}(x) dx = \int_{\alpha_i}^{\beta_i} f_n(x) dx \longrightarrow 0$$

Thus,

$$\langle f_n, g \rangle \longrightarrow 0$$

So $f_n \rightharpoonup 0$.

($\Longrightarrow$):
Theorem: Every weakly convergent sequence in a Banach space X is uniformly bounded.

Proof.
Let $T_n \in X^{**}$ be defined by $T_n(\varphi) = \varphi(x_n)$, for all $\varphi \in X^*$. Then for any $\varphi \in X^*$, $\{\varphi(x_n)\}$ is convergent, following from the weak convergence. Thus, $\{\varphi(x_n)\}$ is bounded. By Uniform Boundedness Principle, we know $\sup_{n\in\mathbb{N}}\|T_n\| < \infty$. Also, $\sup_{n\in\mathbb{N}}\|T_n\| = \sup_{n\in\mathbb{N}}\|x_n\|$, so we conclude that $\{x_n\}$ is uniformly bounded. This is actually the corollary above. □

Then we have finished proving the first part of the claim. Furthermore, for any $a < b$, we have

$$\int_a^b f_n(x)dx = \int_{\mathbb{R}} f_n(x)\chi_{[a,b]}(x)dx = \langle f_n, \chi_{[a,b]}\rangle \to 0 \quad \text{as } n \to \infty$$

Therefore,

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = 0$$

□

2021 May Q2 Find the weak limit of the family $\{f_\epsilon(x)\}$ in $L^2(0,1)$ as $\epsilon \to 0$ for $f_\epsilon(x) = \sin\frac{x}{\epsilon}$. Does the family $\{f_\epsilon(x)\}$ converge strongly in $L^2(0,1)$? Justify your answers.

Proof.
Note that simple functions are dense in $L^2(0,1)$, we may just consider simple functions. Let $g(x) = \sum_{i=1}^N \chi_{[\alpha_i,\beta_i]}(x)$. Then

$$\begin{aligned}\langle f_\epsilon, \chi_{[a,b]}\rangle &= \int_0^1 f_\epsilon(x)\chi_{[a,b]}(x)dx \\ &= \int_a^b f_\epsilon(x)dx = -\epsilon\left(\cos\frac{b}{\epsilon} - \cos\frac{a}{\epsilon}\right) \longrightarrow 0 \quad \text{as } \epsilon \longrightarrow 0\end{aligned}$$

Thus, $\langle f_\epsilon, g\rangle \to 0$ as $\epsilon \to 0$. We conclude that $f \rightharpoonup 0$. Notice that

$$\int_0^1 f_\epsilon^2(x)dx = \int_0^1 \sin^2\frac{x}{\epsilon}dx \longrightarrow \frac{1}{2}$$

So it does not strongly converge to 0. □

2021 Jan Q4 Let H be a Hilbert space. Prove that the following are equivalent:

(i) H is infinite dimensional.

(ii) For every vector $y \in H$ with $\|y\| \le 1$, there exists a sequence of unit vectors $(x_n)_{n\ge1}$ which converges weakly to y.

Proof.
($\Longrightarrow$) : Note that H is infinite dimensional. So for every $y \in H$ with $\|y\| \le 1$, exists an orthonormal sequence $\{e_n\}_{n\in\mathbb{N}}$ such that $\langle y, e_n\rangle = 0$ for all n. Take

$$x_n = y + \sqrt{1 - \|y\|^2}e_n$$

We have

$$\|x_n\|^2 = \|y\|^2 + (1 - \|y\|)^2 = 1 \Longrightarrow \|x_n\| = 1$$

Also, for any $x \in H$, we have

$$\langle x, x_n\rangle = \langle x, y + \sqrt{1 - \|y\|^2}e_n\rangle = \langle x, y\rangle \quad \text{as } n \longrightarrow \infty$$

Here we use the property that every orthonormal sequence weakly converges to 0 in any Hilbert space.

$(\Longleftarrow)$:

Theorem: Weakly convergence sequence implies strong convergence if the sequence is in a compact set.

Proof.
Denote C as the compact set. Suppose $\{x_n\}$ converges weakly to x. We know any sequence in C has a convergent subsequence. Therefore, $\|\times_{n_k} - x'\| \longrightarrow 0$ for some $x' \in C$, which directly leads to $x_{n_k} \rightharpoonup x'$. However, since $\{x_n\}$ weakly converges to x, we know $\{x_{n_k}\}$ weakly converges to x. By the uniqueness of weak limit, we know $x = x'$. Now we can conclude that for every subsequence of $\{x_n\}$, there exists a subsubsequence that stronly converges to x. Therefore, $x_n \longrightarrow x$, as desired. $\square$

Now assume that H is finite dimensional. Then the closed unit ball of H is compact. By this theorem, we directly know that if $x_n \rightharpoonup y$, then $x_n \to y$. So if $\|x_n\| = 1$, we know $\|y\| = 1$. Then for any $\|y\| < 1$, there is no unit vectors sequence such that it weakly converge to y. Thus, H must be infinite dimensional.

$\square$

2020 Aug Q1 Let X be a Banach space and let $(x_n)_{n\in\mathbb{N}}$ and x be in X. Prove that if the sequence (x_n) converges weakly to x then $x \in \overline{\text{Span}\{x_n : n \in \mathbb{N}\}}$.

Proof.
For a convex set in a normed space, we know closure implies weak closure. So $x \in \overline{\text{Span}\{x_n : n \in \mathbb{N}\}}$. One may use Hahn-Banach as well. $\square$

7 2024 May Qualifying Exam

Problem 1 Let $H = \ell^2$ and $\Lambda : H \to H$ be defined by

$$\Lambda x = (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots)$$

where $x = (x_1, x_2, \cdots) \in H$. Prove that Λ is a compact operator and find its spectrum.

Proof.
Define $\Lambda_n = (0, x_1, \frac{x_2}{2}, \cdots, \frac{x_n}{n}, 0, 0, 0, \cdots)$. Obviously, Λ_n is a linear operator with finite rank, so it is a compact operator. Since $\Lambda x \in H$, we know

$$\sum_{k=1}^{\infty} \frac{x_k^2}{k^2} < +\infty \Longrightarrow \forall \varepsilon > 0, \exists N > 0 \text{ s.t. } \forall n > N \Longrightarrow \sum_{k=n}^{\infty} \frac{x_k^2}{k^2} < \varepsilon$$

One may check that

$$\|\Lambda - \Lambda_n\| \longrightarrow 0 \quad \text{as } n \to \infty$$

So Λ is also a compact operator. Then we know that $\sigma(\Lambda) = \sigma_p(\Lambda) \cup \{0\}$. Assume that $\lambda \neq 0$ is an eigenvalue of Λ. Then

$$\Lambda u = \lambda u$$

for some $u \in H$. We have

$$(0, u_1, \frac{u_2}{2}, \cdots) = (\lambda u_1, \lambda u_2, \cdots) \Longrightarrow u_k = 0 \quad \forall k \in \mathbb{N} \Longrightarrow u = 0$$

But eigenvectors cannot be zero, we conclude that λ is not an eigenvalue of Λ. Thus, $\sigma(\Lambda) = \{0\}$.

$\square$

Problem 2 On the Banach space $X = C[0,1]$ of continuous functions with the norm $\|u\|_X = \sup_{x\in[0,1]}|u(x)|$, consider the linear operator $\Lambda : X \to X$ defined by

$$(\Lambda u)(t) = \frac{1}{t}\int_0^t u(x)dx \quad \text{for } t \in (0,1]$$

$$(\Lambda u)(0) = u(0)$$

Prove that $\Lambda : X \to X$ is a continuous and injective operator.

Proof.
We have

$$\|\Lambda u\| = \sup_{t\in[0,1]}|(\Lambda u)(t)| \leq \max\{|u(0)|, \frac{1}{t}\cdot t \cdot L\} = max\{|u(0)|, L\} < \infty$$

where $L = \|u\|$. So Λ is bounded, hence continuous. Suppose $\Lambda u = \Lambda v$ for some $u, v \in C[0,1]$. Then

$$u(0) = v(0)$$

For $t \in (0,1]$, by Fundamental Theorem of Calculus, we have

$$\frac{1}{t}\int_0^t u(x)dx = \frac{1}{t}\int_0^t v(x)dx$$

$$\implies \int_0^t u(x)dx = \int_0^t v(x)dx$$

$$\implies u(t) = v(t)$$

Therefore, $u = v$. We conclude that Λ is injective. $\square$

Problem 3 Let X be a normed space and let $Y \subset X$ be a finite dimensional subspace. Prove that there exists a bounded linear operator $P : X \to Y$ such that $P(y) = y$ for every $y \in Y$.

Proof.
Let $\{u_1, \cdots, u_N\}$ be a unit basis of Y. For any $y \in Y$, we have

$$y = \sum_{i=1}^{N} \alpha_i(y) u_i$$

for some linear functional $\alpha_i : Y \to \mathbb{R}$. Note that $\|u_i\| = 1$, which is bounded. By Hahn-Banach Extensions, we know exists linear functionals $\beta_i : X \to \mathbb{R}$ such that $\|\beta_i\| = \|\alpha_i\|$ and $\beta_i(y) = \alpha_i(y)$ for all $y \in Y$. Define $P : X \to Y$ by

$$P(x) = \sum_{i=1}^{N} \beta_i(x) u_i$$

We have $P(y) = y$ for any $y \in Y$. Furthermore,

$$\|P\| \leq \sum_{i=1}^{N} \|\beta_i\| < \infty$$

as desired. $\square$

Problem 4 Let X be a reflexive Banach space. Consider a sequence $(x_n)_{n\leq 1}$ of points in X such that $\liminf_{n\to\infty}\|x_n\| = \lambda < +\infty$. Prove that there exists a weakly convergent subsequence $x_{n_k} \rightharpoonup x$, with $\|x\| \leq \lambda$.

8 Midterm and Final Exams

This section contains several questions from the midterm & final exams of MATH 503 in 2023-24 Spring.

Problem 1 Let H be a Hilbert space. Let $T : H \to H$ be a bounded linear operator with norm $\|T\| \leq \frac{1}{3}$, prove that $A = I + T$ is injective and has a continuous inverse with $\|A^{-1}\| \leq \frac{3}{2}$.

Proof.
For any $v \in Ker(I+T)$, we have

$$(I+T)v = 0 \Longrightarrow -Tv = v$$

Notice that if $-T$ is a contraction, ie: exists a unique fixed point u such that $-Tu = u$, then $Ker(I+T)$ only contains u. We know $-T$ is a contraction, following from the fact $\|T\| < 1$. So $-T$ only has a fixed point, which is 0. We then know that $Ker(I+T) = \{0\}$, $I+T$ is injective.

For the second part, we recall **Theorem 5.12** in the textbook. To prove $\|A^{-1}\| \leq \frac{3}{2}$, it suffices to show that $\langle Ax, x\rangle \geq \frac{2}{3}\|x\|^2$ for all $x \in H$.

$$\langle Ax, x\rangle \geq \frac{2}{3}\|x\|^2 \Longleftrightarrow \|x\|^2 + \langle Tx, x\rangle \geq \frac{2}{3}\|x\|^2 \Longleftrightarrow \langle Tx, x\rangle \geq \frac{-1}{3}\|x\|^2 \Longleftrightarrow \langle -Tx, x\rangle \leq \frac{1}{3}\|x\|^2$$

which the last inequality is true by Cauchy-Schwarz inequality. Therefore, $\|A^{-1}\| \leq \frac{3}{2}$. □

Problem 2 Let H be a Hilbert space and Y is a closed subspace of H. Prove that every bounded linear operator $T : Y \to H$ can be extended to $T' : H \to H$ defined on the whole space with the same operator norm.

Proof.
We know

$$H = Y \oplus Y^{\perp}$$

Take

$$T' = T \circ P_Y$$

where P_Y is the projection operator from H onto Y. One may check the operator norms by themselves. □

Problem 3 State the Riesz representation theorem. If possible, apply the theorem to the linear functionals

$$\varphi_1(f) = \int_0^{\infty} f(x)dx, \quad \varphi_2(f) = \int_0^1 x^2 f(x)dx$$

on the Hilbert space $H = L^2(\mathbb{R})$. Otherwise, explain why the theorem does not apply.

Proof.
The theorem is not applicable on φ_1 since it is not a bounded functional. For example, take

$$f(x) = \chi_{[1,\infty)}\frac{1}{x}$$

We know $f \in H$ but $\varphi_1(f) = \int_1^\infty \frac{1}{x}dx$ is not integrable.

For φ_2, exists some $g \in H$ such that $\varphi_2(f) = \langle f, g\rangle$ for all $f \in H$. Take $g(x) = \chi_{[0,1]}(x) \cdot x^2$. One may check that $g \in H$. For any $f \in H$, we have

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)g(x)dx = \int_0^1 x^2 f(x)dx = \varphi_2(f).$$

□

Problem 4 Let X be an infinite dimensional Banach space.

(i) If $A : X \to X$ is a compact linear operator, prove that $Range(A) \neq X$.

(ii) Let X_0 be the space of all sequences of real numbers $x = (x_1, x_2, x_3, \cdots, x_N, 0, 0, \cdots)$ with finitely many non-zero entries with norm $\|x\| = \max_k |x_k|$. Consider the compact linear operator $A : X \to X$ defined by

$$Ax = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots, \frac{x_N}{N}, \cdots)$$

Is A injective? Is A surjective? Does this contradict (i)?

Proof.

(i) Suppose $Range(A) = X$. Then A is bounded and surjective. By Open Mapping Theorem, we know A is open. There exists an open ball O such that $\overline{A(O)} = \overline{B_X(0,1)}$, where $B_X(0,1)$ is the unit ball in X. Then $\overline{B_X(0,1)}$ is compact, we reach the contradiction that X is finite-dimensional.

(ii) A is bijective. Since X_0 is not complete, it does not contradict (i).

□

Problem 5 On the space $L^2[0, \pi]$, consider the linear operator

$$(\Lambda f)(x) = \sum_{k=1}^{2024} \left(\int_0^\pi \sqrt{k} \sin(ky) f(y) dy \right) \sin(kx).$$

Prove that Λ is compact and self-adjoint.

Proof.
For any $f \in L^2[0, \pi]$, we know $f \in L^1[0, \pi]$. Let $\{f_n\}$ be a bounded sequence in $L^2[0, \pi]$, then exists $M > 0$ such that $\|f_n\|_1 \leq M$ for all $n \in \mathbb{N}$. For any $x \in [0, \pi]$, we have

$$|(\Lambda f_n)(x)| \leq \sum_{k=1}^{2024} \int_0^\pi |\sqrt{k} \sin(ky) f(y)| dy \sin(kx) \leq 2024^{\frac{3}{2}} M$$

Therefore, $\{\Lambda f_n\}$ is uniformly bounded. Obviously, $\{\Lambda f_n\}$ is also equicontinuous. By Arzela-Ascoli, we know exists $\{\Lambda f_{n_k}\}$ converges. Thus, Λ is compact. Furthermore,

$$\langle \Lambda f, g\rangle = \langle f, \Lambda^* g\rangle \Longrightarrow (\Lambda^* g)(y) = \sum_{k=1}^{2024} \left(\int_0^{\pi} \sqrt{k} \sin(kx) f(x) dx \right) \sin(ky)$$

So we conclude that Λ is self-adjoint. $\square$

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